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Unstable singularities in the σ model†

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Abstract. In the semi-classical approximation to the SU(3) σ model we show how the breakdown of symmetry can be viewed as an unfolding of an unstable singularity in the potential. The examples arising include the Riemann–Hugoniot and parabolic umbilic catastrophes of Thom as well as catastrophes of much larger codimension.

1. Introduction

The σ model (Gell-Mann and Lévy 1960, Lévy 1967) demonstrates both spontaneous and explicit symmetry breaking of a chiral $SU(n)\otimes SU(n)$ invariance. The model has been studied both in the tree graph (semi-classical) approximation (Carruthers and Haymaker 1972 and references therein, McKay *et al* 1973, Hu 1973b) and as a renormalized quantum field theory (Lee 1969, Chan and Haymaker 1973a, b). Starting from a basic chiral invariant model, one attempts to reach by perturbation methods a broken symmetry physical model retaining only a residual symmetry of isospin, parity and hypercharge. Studies in the tree graph approximation show how such calculations can be spoiled by the occurrence of singularities that limit convergence of a perturbation expansion well before physical values of the expansion parameters are reached (Carruthers and Haymaker 1971a). Such studies indicate also that the order in which one turns on the symmetry breaking parameters is important if one is to reach a physically reasonable model from the underlying symmetric starting point without encountering a singularity (Carruthers and Haymaker 1971b).

Our aim here is to look at the $SU(3)\otimes SU(3)$ σ model in the semi-classical approximation and to show how the problem of symmetry breakdown fits closely into the mathematical theory of singularities of smooth maps. We will demonstrate how the breakdown of symmetry in this model can be viewed as the unfolding of an unstable but finitely determined map germ with certain Lagrangian parameters playing the role of control variables and the vacuum expectation value of scalar fields serving as state variables. The difficulties met with in turning on symmetry breaking by perturbation theory constitute the phenomenon called a catastrophe by Thom (1972). In the $SU(3)\otimes SU(3)$ σ model we shall see examples of Thom's Riemann–Hugoniot and parabolic umbilic catastrophes as well as the possibility of catastrophes of much larger codimension.

In what follows, we will try to point out, without too much detail, how several physical concepts in the semi-classical approximation to the σ model parallel closely the mathematical ideas of singularity theory. For all details of singularity theory we refer

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the reader to the excellent lecture notes of Arnol'd (1968) and Wall (1971) where clear expositions of, and references to, the work of Boardman and Mather may be found. For examples of the physical application of the σ model one may consult any of the references already mentioned.

2. Basic concepts

The SU(3) ⊗ SU(3) form of the σ model is constructed with eighteen fields transforming as members of the $(3, \bar{3}) \oplus (\bar{3}, 3)$ representation of SU(3)⊗SU(3) (Lévy 1967, Carruthers and Haymaker 1972 and references therein). These fields, a scalar nonet σ_i and a pseudoscalar nonet ϕ_i, are simply described by a three by three complex matrix $\mathcal{M}$,

$$\mathcal{M} = \sum_{j=0}^{8} \left(\frac{\sigma_j + \mathrm{i}\phi_j}{\sqrt{2}}\right) \lambda^j, \tag{2.1}$$

with the usual SU(3) matrices λ^j. Under chiral transformation $\mathcal{M}$ changes to $U\mathcal{M}V^\dagger$ (U, V unitary matrices) so that the basic invariant Lagrangian density for the model can be written in the form

$$L = \tfrac{1}{2} \operatorname{Tr} \partial_\mu \mathcal{M}^\dagger \partial^\mu \mathcal{M} + f_1 (\operatorname{Tr} \mathcal{M}^\dagger \mathcal{M})^2 + f_2 \operatorname{Tr}(\mathcal{M}^\dagger \mathcal{M})^2 + g(\det \mathcal{M} + \det \mathcal{M}^\dagger), \tag{2.2}$$

with f_1, f_2, g as coupling parameters. Writing $L = T - V$ we have a potential function V defined by

$$\begin{aligned} V &= -f_1 (\operatorname{Tr} \mathcal{M}^\dagger \mathcal{M})^2 - f_2 \operatorname{Tr}(\mathcal{M}^\dagger \mathcal{M})^2 - g(\det \mathcal{M} + \det \mathcal{M}^\dagger) \\ &= V(\sigma_0, \ldots, \sigma_8, \phi_0, \ldots, \phi_8). \end{aligned} \tag{2.3}$$

In the semi-classical approximation we regard V as a classical potential function to be minimized with respect to its field variables. In this approximation the vacuum expectation values of the fields are given by the value of the field variables at which the extremum occurs. The squares of the particle masses are obtained in this model as coefficients of the quadratic terms in the Taylor series expansion of V about its minimum. Higher-order terms in such a series expansion contribute to interactions among the fields, but we will not consider these interaction terms in the present discussion.

The potential V is SU(3) ⊗ SU(3) invariant but for a more realistic model we wish to preserve only a residual symmetry of isospin, hypercharge and parity. Therefore, although V clearly has an extremum when all fields vanish, $\sigma_i = \phi_i = 0$, we will look for additional extrema at points where $\phi_i = 0$, $i = 0, 1, \ldots, 8$, $\sigma_j = 0$, $j = 1, 2, \ldots, 7$, but $\sigma_0 \neq 0$, $\sigma_8 \neq 0$. Such extrema spontaneously break the chiral symmetry. Moreover, we have the freedom to modify V by adding terms such as $\epsilon_0\sigma_0 + \epsilon_8\sigma_8$ which explicitly break the symmetry. Among the many different forms which we might choose for V, for simplicity we will restrict ourselves by only allowing explicit symmetry breaking terms which belong to the $(3, \bar{3}) \oplus (\bar{3}, 3)$ representation of SU(3) ⊗ SU(3) (Gell-Mann *et al* 1968).

Since we are interested in extrema of V at which $\sigma_0 \neq 0$, $\sigma_8 \neq 0$ but all other fields vanish, define a reduced potential $v(\sigma_0, \sigma_8)$ by

$$v(\sigma_0, \sigma_8) = V(\sigma_i, \phi_j)|_{\phi_0 = \phi_1 = \ldots = \phi_8 = \sigma_1 = \sigma_2 = \ldots = \sigma_7 = 0}. \tag{2.4}$$

The field vacuum expectation values $\xi_0 = \langle\sigma_0\rangle$, $\xi_8 = \langle\sigma_8\rangle$ are now determined by the values of σ_0, σ_8 at which v has an extremum, while all other fields have zero vacuum expectation values. An expansion of v about the extremum gives us not only masses for the scalar fields σ_0, σ_8, but, by inserting the expansion into V as well, we induce mass contributions for the other sixteen fields as well. It is clear from (2.3) that both V and v have unstable singularities (coincident extrema) at $\phi_i = \sigma_i = 0$, $i = 0, \ldots 8$. For sixteen fields the residual symmetry requirements constrain the coincident extrema to remain coincident at zero field strength. However, we may allow the unstable coincident singularities at $\sigma_0 = \sigma_8 = 0$ to split and move away from the origin to give breakdown of the chiral symmetry. That is to say, we allow the unstable singularity of v at the origin to unfold.

Next we must discuss the notion of stability. The physical stability of the σ model in the semi-classical approximation simply means that the potential must be bounded below and that all squared particle masses should be positive definite. On the other hand, there are several mathematical notions of stability of a map at a point. Mather develops the mathematical idea of stability with respect to various groups of diffeomorphisms which act on certain function spaces (see Wall 1971). The reduced potential v above is an example of a function on which these different groups might act. The simplest group $\mathscr{R}$ acts on v by diffeomorphically changing coordinates in the source space of v. We may also change coordinates in the target space of v by a group $\mathscr{L}$. We may do both together by a product group $\mathscr{A} = \mathscr{R} \otimes \mathscr{L}$, or even by a semi-direct product $\mathscr{K}$, in which changes in the target space depend on position in the source space. $\mathscr{R}$ gives the weakest notion of stability and $\mathscr{K}$ the strongest. Which of these notions is suitable to the physical context of the σ model?

To interpret physically a field theory model we need to be able to make power series expansions of the potential in terms of its field variables. However, the equivalence theorem (Coleman *et al* 1969) tells us that the physical content of the field theory is not altered if we re-define the field variables by analytic diffeomorphisms. Such field re-definition is done by elements of the group $\mathscr{R}$ above, hence $\mathscr{R}$-stability is the mathematical concept of stability suitable for the σ model. Although Mather's analysis deals with C^∞ diffeomorphisms, we need here only the simpler concept of analytic stability under the group $\mathscr{R}$. The mathematical problem posed by the σ model can now be simply stated. Regarding $v(\sigma_0, \sigma_8)$ as a map germ analytically unstable at $\sigma_0 = \sigma_8 = 0$, determine its codimension (with respect to $\mathscr{R}$) in the space of analytic map germs and construct unfoldings.

A word of technical caution is in order at this point. We are grateful to Professor Wall (private communication) for pointing out the following results to us: (i) the codimension of a germ can be different under the different groups mentioned above; (ii) however, $\operatorname{codim}_{\mathscr{R}} v < \infty$ if and only if $\operatorname{codim}_{\mathscr{K}} v < \infty$; (iii) in general $\operatorname{codim}_{\mathscr{K}} v \leq \operatorname{codim}_{\mathscr{R}} v$; (iv) for homogeneous germs $\operatorname{codim}_{\mathscr{K}} v = \operatorname{codim}_{\mathscr{R}} v$; (v) all codimensions less than or equal to six are the same. In the examples given in § 4 we often found it much easier to compute codimensions using $\mathscr{K}$ rather than $\mathscr{R}$. The points listed above helped us to relate $\operatorname{codim}_{\mathscr{K}}$ to $\operatorname{codim}_{\mathscr{R}}$. In one of these examples we have only computed $\operatorname{codim}_{\mathscr{K}}$ and not $\operatorname{codim}_{\mathscr{R}}$.

3. Unfoldings

We now need some notation. By X we denote an n-dimensional Euclidean space with coordinates (state variables) $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$. By S denote an N-dimensional

Euclidean space with coordinates (control variables) $\boldsymbol{s}=(s_1, s_2, \ldots, s_N)$. Let $M=(X, S)$ be the Cartesian product of X and S and let P be a Euclidean space with coordinates $\boldsymbol{y}=(y_1, y_2, \ldots, y_{N+1})$. Let $v(x_1, \ldots, x_n)$ be the germ of a map from X to $\mathbb{R}$ unstable at the origin of X but of finite codimension N. According to Mather's analysis (see Wall 1971) choose N basis elements $f_1(\boldsymbol{x}), f_2(\boldsymbol{x}), \ldots, f_N(\boldsymbol{x})$ for the unfolding of v, and define the map $H: M \rightarrow P$

$$y_1 = v(\boldsymbol{x}) + \sum_{j=1}^{N} s_j f_j(\boldsymbol{x}) \qquad y_2 = s_1$$
$$y_i = s_{i-1} \qquad y_{N+1} = s_N. \tag{3.1a}$$

At the same time we will be interested in the map $h: X \rightarrow \mathbb{R}$ defined by

$$y_1 = v(\boldsymbol{x}) + \sum_{j=1}^{N} s_j f_j(\boldsymbol{x}), \tag{3.1b}$$

where $s_1, \ldots, s_N$ are regarded as fixed. Either (3.1a) or (3.1b) can be viewed as an unfolding of v (Thom 1972, Wall 1971). The map H 'lifts' the unstable singularity of v at $\boldsymbol{x}=0$ to a stable singularity of H at $\boldsymbol{x}=0, \boldsymbol{s}=0$. Clearly the first differential of H, $\mathrm{d}H$,

d^2H so that Σ^n may be subdivided into regions each of which corresponds to fixed index and rank of this quadratic form. Such regions are closely related to the Boardman manifolds $\Sigma^{n,j}$, $0 \leq j \leq n$ (Arnol'd 1968).

By H', denote the restriction of H to the manifold Σ^n,

$$H' = H|\Sigma^n. \tag{3.4}$$

Then we may regard $\Sigma^{n,j}$, $0 \leq j \leq n$, as the set of points of Σ^n at which $\ker dH'$ has dimension j (Arnol'd 1968). To see what this means in the σ model context, suppose we are at a point $(\boldsymbol{x}_0, s_0)$ of Σ^n which is so situated that we may lift the variables $s_1, \ldots, s_N$ to serve as coordinates in the tangent space to Σ^n at $(\boldsymbol{x}_0, s_0)$, $T\Sigma^n$. From (3.1a) and (3.4) it is clear that dH' is of rank N at $(\boldsymbol{x}_0, s_0)$ and $\ker dH'$ is void, hence the point $(\boldsymbol{x}_0, s_0)$ lies in the manifold $\Sigma^{n,0}$. In other words $\Sigma^{n,0}$ consists of the points of Σ^n at which we may identify TS with $T\Sigma^n$ (using the s_i as coordinates in both S and TS) to get

$$TM = TX + T\Sigma^n.$$

The points in $\Sigma^{n,0}$ are those points in Σ^n at which the tangent spaces TX and $T\Sigma^n$ are transverse. Consequently the manifolds $\Sigma^{n,j}$, $j \neq 0$, consist of points of Σ^n at which TX and $T\Sigma^n$ are not transverse and have an intersection $TX \cap T\Sigma^n$ of non-zero dimension. At such a point in $\Sigma^{n,j}$, we may choose at least one coordinate x_1 in $TX \cap T\Sigma^n$. We then can choose local coordinates $(x_1, s_2', \ldots, s_N')$ on Σ^n and $(x_1, x_2, \ldots, x_n)$ on X. Since by (3.2) $\text{grad}_x\, y_1$ vanishes identically on Σ^n, we see that

$$\partial^2 y_1/\partial x_1 \partial x_j = 0$$

at this point for $j = 1, 2, \ldots n$. Depending on the dimension of $TX \cap T\Sigma^n$ there may be up to n such coordinates x_i. We conclude that $\Sigma^{n,j}$ contains those points of Σ^n at which we may choose j local coordinates $x_1, \ldots, x_j$ in $T\Sigma^n$ such that

$$\partial^2 y_1/\partial x_i \partial x_k = 0, \qquad i = 1, 2, \ldots, j, \qquad k = 1, \ldots, n.$$

Comparing this with (3.3) we can say, in physical terms, that $\Sigma^{n,j}$ is the locus of points on Σ^n at which j particle masses associated with the scalar fields $x_1, \ldots, x_j$ vanish.

For physical stability of the σ model we desire the state variables (fields) to lie only in that portion of Σ^n in which all squared particle masses are positive definite. If we denote this region of physical stability by $\bar{\Sigma}^n$, we observe that $\bar{\Sigma}^n$ is a subset of $\Sigma^{n,0}$ bounded by $\Sigma^{n,1}$ which we must enter to make at least one squared scalar mass vanish and so become negative. The transversality argument above concerning TX and $T\Sigma^n$ also shows that if the state variables x_i have branch points as functions of the control variables s_j, then such branch points can occur only at 'folds' of Σ^n, i.e., only at points lying in some $\Sigma^{n,j}$ for $j \neq 0$. In physical terms, such branch points can occur only at points where at least one scalar mass vanishes as was first pointed out by McKay and Palmer (1972).

4. Examples

We now illustrate the general considerations above by specific examples based on the σ model potential function (2.3). In order that V be bounded from below for large values of the fields we must require that

$$\begin{aligned} f_1+f_2 &\leqslant 0, \qquad \text{for } f_1<0, \\ 3f_1+f_2 &\leqslant 0, \qquad \text{for } f_1 \geqslant 0. \end{aligned} \tag{4.1}$$

Bearing in mind (4.1) we start with a simple example which can be visualized readily to illustrate the concepts of the previous sections. Let σ_8 be held fixed at the origin, and consider v to depend only on the variable $x_1 = \sigma_0$. In addition set $g = 0$, corresponding to scale invariance of the model (Carruthers and Haymaker 1971b). From (2.3) and (2.4) we have

$$v_1(x_1) = -(f_1 + \tfrac{1}{3}f_2)x_1^4. \tag{4.2a}$$

The germ v_1 has a simple unstable singularity at $x_1 = 0$ consisting of three coincident extrema. One readily calculates $\text{codim}_{\mathscr{R}}\, v_1 = 2$, with unfolding h given by

$$y_1 = -(f_1 + \tfrac{1}{3}f_2)x_1^4 + s_1x_1 + s_2x_1^2. \tag{4.2b}$$

This is Thom's (1972) Riemann–Hugoniot catastrophe. The singularity surface Σ^1 of the map H corresponding to h of (4.2b) is defined by

$$\text{grad}_{x_1}\, y_1 = -4(f_1 + \tfrac{1}{3}f_2)x_1^3 + s_1 + 2s_2x_1 = 0. \tag{4.2c}$$

The squared mass m_0^2 of the field σ_0 is given by (3.3) as

$$m_0^2 = \partial^2 y_1/\partial x_1^2 = -12(f_1 + \tfrac{1}{3}f_2)x_1^2 + 2s_2. \tag{4.2d}$$

The unfolding moves the coincident extrema apart, either into the complex plane ($s_2 > 0$) or into two additional real extrema ($s_2 < 0$). The singularity surface Σ^1 is a ruled surface with a pair of folds for $s_2 < 0$. On Σ^1, m_0^2 can be both positive and negative. The condition $m_0^2 = 0$ defines $\Sigma^{1,1}$, a one-dimensional manifold, which splits $\Sigma^{1,0}$ into two connected parts, one with positive m_0^2, one with negative m_0^2. $\Sigma^{1,1}$ coincides with the fold lines of Σ^1 and one can readily see that along $\Sigma^{1,1}$, the tangent plane to Σ^1 is not transverse to the x_1 axis. In physical terms we can realize this unfolding by adding to the original potential V in (2.3) the terms $\frac{1}{2}\mu^2\, \text{Tr}(\mathscr{M}^\dagger \mathscr{M}) + \epsilon_0\sigma_0$. The bare mass term μ^2 is chiral invariant while the linear term in ϵ_0 explicitly breaks the symmetry. Nevertheless μ^2 and ϵ_0 are on the same footing as control parameters stabilizing the singularity of v_1.

Now modify the example slightly by allowing the cubic interaction term to be present initially with $g \neq 0$. Then we have

$$v_2(x_1) = -(f_1 + \tfrac{1}{3}f_2)x_1^4 + (2g/3\surd 3)x_1^3. \tag{4.3a}$$

The germ v_2 differs from v_1 because the presence of the cubic term has already moved one extremum of v_1 a finite distance away from $x_1 = 0$. The origin is still an unstable singularity, but it is unstable like x_1^3 rather than like x_1^4 as in v_1. Hence $\text{codim}_{\mathscr{R}}\, v_2 = 1$, with unfolding

$$y_1 = v_2(x_1) + s_1x_1. \tag{4.3b}$$

This example emphasizes the point that the type of instability and the manner of unfolding depend strongly on our choice of the initial unstable germ v. Speaking physically we might say that the type of instability depends on what we consider to be the *a priori* underlying chiral invariant theory.

Now look at non-trivial symmetry breaking involving the two fields σ_0, σ_8. Let us do the stability analysis in terms of the simple variables

$$x_1 = \frac{1}{\sqrt{3}}(\sqrt{2}\sigma_0 + \sigma_8),$$
$$x_2 = \frac{1}{\sqrt{3}}(\sigma_0 - \sqrt{2}\sigma_8). \tag{4.4}$$

Assuming f_1, f_2, g to be all non-zero constants given initially (consistent with (4.1)), we write the reduced potential as

$$v_3(x_1, x_2) = -(f_1 + \tfrac{1}{2}f_2)x_1^4 - (f_1 + f_2)x_2^4 - 2f_1x_1^2x_2^2 - gx_1^2x_2. \tag{4.5a}$$

The stability analysis of v_3 is done most simply in terms of the group $\mathscr{K}$. We find that if $f_1 + f_2 \neq 0$, $\text{codim}_{\mathscr{K}}\, v_3 = 4$, and hence $\text{codim}_{\mathscr{R}}\, v_3 = 4$ by the remarks above at the end of § 2. The unfolding can be written as

$$y_1 = v_3(x_1, x_2) + s_1x_1 + s_2x_2 + s_3x_1^2 + s_4x_2^2. \tag{4.5b}$$

By a theorem of Lu (1970) there exists an analytic change of coordinates that converts (4.5*b*) into the standard form of Thom's (1972) parabolic umbilic. If $f_1 + f_2 = 0$, v_3 has infinite codimension and there is no unfolding. From (4.1) we note that this situation corresponds to the boundary of the region of physical stability. It is clear intuitively why no classification is possible for this case. If $f_1 + f_2 = 0$, v_3 contains x_1^2 as a common factor in all its terms and hence the extremum at the origin is not isolated but lies on a line of extrema given by $x_1 = 0$. The breakdown of the mathematical classification for a non-isolated extremum is reflected in the physical ambiguity of a potential with a non-isolated minimum.

For v_3, as for v_2, the cubic interaction terms ($g \neq 0$) reduce the instability at the origin. Thus let us demand $g = 0$ from the beginning. We have

$$v_4(x_1, x_2) = -(f_1 + \tfrac{1}{2}f_2)x_1^4 - (f_1 + f_2)x_2^4 - 2f_1x_1^2x_2^2, \tag{4.6}$$

where f_1, f_2 are assumed as fixed constants. Since v_4 is homogeneous the $\mathscr{K}$ and $\mathscr{R}$ analyses are the same. We note first that v_4 has infinite codimension in the cases $f_2 = 0$, $f_1 + f_2 = 0$, $f_1 + \frac{1}{2}f_2 = 0$, $f_1 + \frac{1}{3}f_2 = 0$. As for v_3, these exceptional cases represent instances where the extremum at the origin is not isolated. Excluding these exceptional cases we find $\text{codim}_{\mathscr{R}}\, v_4 = 8$. Basis functions for an unfolding can be chosen to be x_1, x_2, x_1^2, x_1x_2, x_2^2, $x_1x_2^2$, $x_1^2x_2$, and $x_1^2x_2^2$. Note the curious fact that the cubic interaction term $x_1^2x_2$ of v_3 here reappears but as a basis element in the unfolding of v_4. Moreover, one quartic term has appeared. Instead of using $x_1^2x_2^2$, an equivalent basis element is $(x_1^2 + x_2^2)^2$ which is just the part of v_4 with coupling strength f_1. Although f_1 was assumed fixed to begin with, it must be varied in order to stabilize v_4. From the viewpoint of v_4, both f_1 and g should be control parameters.

The most economic choice of initial unstable germ is

$$v_5(x_1, x_2) = -\tfrac{1}{2}f_2(x_1^4 + 2x_2^4), \tag{4.7a}$$

with f_2 fixed and non-zero. The germ v_5 has an isolated unstable extremum at the origin with $\text{codim}_{\mathscr{R}}\, v_5 = 8$ and unfolding

$$y_1 = v_5(x_1, x_2) + s_1x_1 + s_2x_2 + s_3x_1^2 + s_4x_2^2 + s_5x_1x_2 + s_6x_1^2x_2 + s_7x_1x_2^2 + s_8(x_1^2 + x_2^2)^2. \tag{4.7b}$$

As a final example we allow spontaneous breaking of isospin invariance by looking for an extremum at $\sigma_0 \neq 0$, $\sigma_8 \neq 0$, $\sigma_3 \neq 0$ (Hu 1973a). If we define

$$x_1 = \frac{1}{\sqrt{3}}(\sqrt{2}\sigma_0 + \sqrt{3}\sigma_3 + \sigma_8),$$

$$x_2 = \frac{1}{\sqrt{3}}(\sqrt{2}\sigma_0 - \sqrt{3}\sigma_3 + \sigma_8), \tag{4.8}$$

$$x_3 = \frac{1}{\sqrt{3}}(\sigma_0 - \sqrt{2}\sigma_8),$$

we have a reduced potential

$$v_6(x_1, x_2, x_3) = -\tfrac{1}{4}(f_1 + f_2)(x_1^4 + x_2^4 + 4x_3^4) - \tfrac{1}{2}f_1(x_1^2x_2^2 + 2x_1^2x_3^2 + 2x_2^2x_3^2) - gx_1x_2x_3. \tag{4.9}$$

With f_1, f_2, and g fixed and non-zero, we find $\operatorname{codim}_{\mathscr{K}} v_6 = 9$. We have not calculated the codimension with respect to $\mathscr{R}$. A basis for a $\mathscr{K}$-unfolding is given by x_1, x_2, x_3, x_1x_2, x_1x_3, x_2x_3, x_1^2, x_2^2, x_3^2. If as in v_5 we assume $f_1 = g = 0$ and start with only $f_2 \neq 0$, then

$$v_7(x_1, x_2, x_3) = -\tfrac{1}{4}f_2(x_1^4 + x_2^4 + 4x_3^4) \tag{4.10}$$

is homogeneous with $\operatorname{codim}_{\mathscr{R}} v_7 = 26$. A point of interest here is that some basis elements for an unfolding are of higher order than v_7 itself, namely $x_1^2x_2^2x_3$, $x_1^2x_2x_3^2$, $x_1x_2^2x_3^2$, and $x_1^2x_2^2x_3^2$.

5. Discussion

We hope it is now clear how the familiar physical ideas of the σ model illustrate many of the mathematical ideas associated with unstable singularities. A striking aspect of the examples above is that most of the control parameters in the unfoldings can be realized naturally in the σ model by adding suitable terms to the initial potential V in (2.3). Thus for v_3, if we add to V the terms $\frac{1}{2}\mu^2 \operatorname{Tr}(\mathscr{M}^\dagger\mathscr{M}) + \epsilon_0\sigma_0 + \epsilon_8\sigma_8$, then μ^2, ϵ_0, ϵ_8 realize three of the four control variables $s_1, \ldots, s_4$ in (4.5*b*). However, by keeping the bare mass term chiral invariant we get $s_3 = s_4 = \frac{1}{6}\mu^2$ and thus constrain one dimension of the unfolding. The work of McKay *et al* (1973) illustrates the potential v_5. They add two further terms to V, $\beta_0U_0 + \beta_8U_8$, where

$$U_i = \frac{\partial}{\partial\sigma_i}(\det\mathscr{M} + \det\mathscr{M}^\dagger). \tag{5.1}$$

They remain within the $(3, \bar{3}) \oplus (\bar{3}, 3)$ model of symmetry breaking but with (5.1) they can in principle realize seven of the eight control variables in (4.7*b*) through the Lagrangian parameters f_1, g, μ^2, ϵ_0, ϵ_8, β_0, β_8. However, one still has the constraint $s_7 = 0$ in their model. One can realize s_7 within the same scheme by adding a cubic term to the potential, $\alpha_0W_0 + \alpha_8W_8$,

$$W_i = \frac{\partial}{\partial\sigma_i}(\operatorname{Tr}\mathscr{M}^\dagger\mathscr{M})^2, \tag{5.2}$$

where one of the two degrees of freedom α_0, α_8 combines with g and the other gives s_7. Thus the unfolding (4.7*b*) can be realized in principle within the σ model although for physical reasons one would normally constrain the unfolding by requiring $\epsilon_8/\epsilon_0 = \beta_8/\beta_0 = \alpha_8/\alpha_0$.

From the mathematical ideas sketched above we note two conclusions of physical importance. Firstly, the codimension of the potential germ under the group $\mathscr{R}$ gives an upper limit to the number of degrees of freedom which should come into play in a symmetry breakdown. So far as physical constraints allow, all these dimensions of unfolding should be realized and explored in a physical model. Secondly, essentially all Lagrangian parameters should be viewed as control variables whether they multiply invariant interaction terms or terms which explicitly break the symmetry. One should therefore feel free to vary all of them in turning on or off the symmetry breaking (Carruthers and Haymaker 1971b).

Finally we must note that in some respects field theory models like the σ model do not at all parallel the mathematical singularity theory. For example, the unfolding of v_7 above produced terms of fifth and sixth order in fields which must be rejected in a renormalizable quantum field theory. Because our potential V retained a residual symmetry, we could only examine the singularity of $v(\sigma_0, \sigma_8)$ while ignoring the dependence on the other sixteen fields. Although the Boardman manifolds $\Sigma^{n,j}$ are related to the question of mass positivity for the σ_0, σ_8 fields, the sixteen other fields must also have positive squared masses which imposes a still finer partition of the singularity surface Σ^n (Okubo and Mathur 1970a, b). In spite of these differences we may still hope that further developments in singularity theory will give added insight on the phenomenon of symmetry breakdown.

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